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COMMENT

Comment on 'invariants of differential equations defined by vector fields'

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Abstract
In Ndogmo (2008 *J. Phys. A: Math. Theor.* **41** 025207), the author claims to have determined a complete list of functionally independent differential invariants up to order 2 for the equivalence group of differential equations defined by vector fields when there are two and three independent variables. In this comment, we show that this is not the case. Using the equivariant moving frame method, we derive a complete set of functionally independent differential invariants of orders 1 and 2 for an arbitrary number $n \geqslant 2$ of independent variables. In the particular case $n = 2$, we obtain six functionally independent invariants, two of which were not found in Ndogmo (2008 *J. Phys. A: Math. Theor.* **41** 025207). In the case $n = 3$, we get 21 functionally independent invariants, six of which are new. We also give a complete classification of the differential invariants.

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1. Introduction

In 1998 and 1999, Fels and Olver developed the theory of equivariant moving frames for Lie groups [3, 4]. Recently, their work has been generalized to Lie pseudo-groups [7–9]. As for finite dimensional Lie groups, the equivariant moving frame method for Lie pseudo-groups gives all functionally independent differential invariants of a pseudo-group action, and establishes the recurrence relations between invariantly differentiated invariants and normalized invariants, [8, 9]. It can also be used to derive the structure equations of Lie pseudo-groups, [7]. The first extensive application of this new theory can be found in [1, 2]. In those two papers, the structure equations for the symmetry pseudo-group of the Kadomtsev–Petviashvili equation and the classification of the differential invariants is carried out in complete detail.

The computation of differential invariants of a symmetry group using Lie's approach requires the integration of a linear system of partial differential equations [6]. With the equivariant moving frame method, differential invariants are derived using only differentiation and solving algebraic equations. Since algebraic equations are usually easier to solve than differential equations, the equivariant moving frame approach frequently gives the differential invariants with less work compared to Lie's approach. This is particularly true for the problem we are concerned with in this paper. In [5], the author uses Lie's approach to derive some of the differential invariants of orders 1 and 2 for the equivalence pseudo-group of transformations for differential equations defined by a vector field

$$\sum_{i=1}^{n} a^i(x)\partial_{x^i}u(x) = 0, \qquad x = (x_1, \ldots, x_n), \tag{1}$$

when there are two and three independent variables. The derivation takes a total of about seven pages. With the equivariant moving frame method the same computation, for an arbitrary number $n \geqslant 2$ of independent variables, is done in one page.

The purpose of this paper is not just to illustrate how the equivariant moving frame method can reduce the amount of computations when it comes to determining the differential invariants of a Lie pseudo-group, but to also address some errors found in [5]. In theorem 6 of [5], the author claims to have found all differential invariants, up to order 2, for the pseudo-group of equivalence transformations of (1) when there are two and three independent variables, but we show that this is not the case. Using the equivariant moving frame technic we establish a complete set of functionally independent differential invariants of (1), up to order 2, for an arbitrary number $n \geqslant 2$ of independent variables. In the cases $n = 2, 3$ we show that we have found more functionally independent invariants than in [5]. Furthermore, the conjecture of [5] on page 12 is false. The conjecture is based on the wrong number of functionally independent invariants of orders 1 and 2 derived by the author when $n = 2, 3$, and on an unjustified quantity W_n at the top of page 12. The conjecture must be replaced by our proposition 5.

The paper is divided as follows. In section 2 we start by giving an outline of the equivariant moving frame theory, then in section 3 we summarize the equivalence problem for the differential equation (1) discussed in [5]. Using the method of equivariant moving frames we derive all functionally independent invariants up to order 2 for the equivalence pseudo-group in section 4, and we finish the paper by studying the 'algebra' of differential invariants.

2. Equivariant moving frame theory

For a detailed exposition of the equivariant moving frame theory for Lie pseudo-groups we refer the reader to [8, 9]. In this section we state the important results without proofs.

2.1. Normalized invariants

Let M be a smooth manifold of dimension m. Let $J^n(M, p)$ be nth jet bundle of equivalence classes of p-dimensional submanifolds S of M with nth order contact. We choose local coordinates on M

$$z = (z^1, \ldots, z^m) = (x^1, \ldots, x^p, u^1, \ldots, u^q) = (x, u), \qquad p + q = m,$$

so that the submanifold S can be expressed as the graph of smooth functions $u^\alpha = f^\alpha(x), \alpha = 1, \ldots, q$. Local coordinates on $J^n(M, p)$ are given by $j^n_z S = (x, u^{(n)})$, where $u^{(n)}$ denotes all derivatives u with respect to the variables x up to order n. We denote by $\mathcal{D}(M)$ the pseudo-group of all local diffeomorphisms of M. For $0 \leqslant n \leqslant \infty$, let $\mathcal{D}^{(n)} \to M$ be the subbundle of $J^n(M, M)$ consisting of the nth order jets $j^n\psi$ of local diffeomorphisms $\psi : M \to M$. Local coordinates on $\mathcal{D}^{(n)}$ are given by $j_z^n\psi = (x, u, X^{(n)}, U^{(n)})$, where $z = (x, u) \in M$ are the source coordinates, $Z = (X, U) \in M$ the target coordinates and $X_A^i = \partial^{\#A} X^i/\partial z^A, i = 1, \dots, p, U_A^\alpha = \partial^{\#A} U^\alpha/\partial z^A, \alpha = 1, \dots, q, 1 \leqslant \#A \leqslant n$, are the derivatives of the target coordinates with respect to the source coordinates. The jet coordinates X_A^i, U_A^α are to be viewed as representing the group parameters of the diffeomorphism pseudo-group $\mathcal{D}$. The right action of $\mathcal{D}$ on $\mathcal{D}^{(n)}$ is defined by

$$R_\psi\left(j_z^n\phi\right) = j_{\psi(z)}^n(\phi \circ \psi^{-1}),$$

when the composition $\phi \circ \psi^{-1}$ is defined.

Definition 1. *A Lie pseudo-group $\mathcal{G}$ is a sub-pseudo-group of $\mathcal{D}(M)$ whose diffeomorphisms are local solutions of an involutive system of defining partial differential equations*

$$F(x, u, X^{(n)}, U^{(n)}) = 0. \tag{2}$$

The pseudo-group $\mathcal{G}$ acts on the submanifold jet bundle $J^n(M, p)$ by mapping the submanifold jet $j_z^n S = (x, u^{(n)})$ to the target jet $\psi|_z \cdot j_z^n S = j_{\psi(z)}^n \psi(S) = (X, \widehat{U}^{(n)}), \psi \in \mathcal{G}$. A hat is added over the transformed jet coordinates to distinguish them from the diffeomorphism jet coordinates U_A^α. The local expressions for $\widehat{U}^{(n)}$ are given by

$$\widehat{U}_J^\alpha = D_{X^{j_1}} \cdots D_{X^{j_k}} U^\alpha, \qquad 0 \leqslant k = \#J \leqslant n, \qquad \alpha = 1, \dots, q, \tag{3}$$

where

$$D_{X^i} = \sum_{j=1}^p W_i^j D_{x^j}, \qquad \text{with} \qquad \left(W_i^j\right) = (D_{x^j} X^i)^{-1}, \qquad i = 1, \dots, p, \tag{4}$$

and D_{x^i} is the total differential operator with respect to x^i.

The pseudo-group jet $\mathcal{G}^{(n)}$ and the submanifold jet $J^n(M, p)$ are put together in the bundle $\mathcal{H}^{(n)} \to J^n(M, p)$ obtained by taking the pull-back of $\mathcal{G}^{(n)} \to M$ along the usual jet projection $\pi^n : J^n(M, p) \to M$. The local coordinates on $\mathcal{H}$ are given by the pair of jets $\left(j_z^n S, j_z^n\phi\right), S \subset M, \phi \in \mathcal{G}$. The pseudo-group $\mathcal{G}$ acts on $\mathcal{H}$ by

$$\psi|_z \cdot \left(j_z^n S, j_z^n\phi\right) = \left(j_{\psi(z)}^n \psi(S), j_{\psi(z)}^n(\phi \circ \psi^{-1})\right). \tag{5}$$

From (5) it follows that the target jet coordinates $\psi|_z \cdot j_z^n S = (X, U^{(n)})$ are invariant under the action of $\mathcal{G}$.

Definition 2. *An nth order moving frame for a pseudo-group $\mathcal{G}$ acting on p-dimensional submanifolds of M is a $\mathcal{G}$-equivariant local section $\rho^{(n)} : J^n(M, p) \to \mathcal{H}^{(n)}$.*

The $\mathcal{G}$-equivariance of the section means that

$$\rho^{(n)}(\psi|_{(x,u)} \cdot (x, u^{(n)})) = \rho^{(n)}(x, u^{(n)})\left(j_{(x,u)}^n\psi\right)^{-1}, \qquad \psi \in \mathcal{G},$$

when all products are defined.

An nth order moving frame exists in a neighborhood of a jet $(x, u^{(n)})$ if and only if $\mathcal{G}$ acts locally freely at $(x, u^{(n)})$ and the action is regular. In applications, a moving frame $\rho^{(n)}$ is obtained in three steps. First compute the prolonged pseudo-group action (3):

$$(X, \widehat{U}^{(n)}) = P^{(n)}(x, u^{(n)}, g^{(n)}), \tag{6}$$

which will depend on r_n pseudo-group parameters $g^{(n)}$. Then set r_n of the coordinate functions (6) to be constant valued

$$P_\nu(x, u^{(n)}, g^{(n)}) = c_\nu, \qquad \nu = 1, \dots, r_n, \tag{7}$$

so as to form a cross-section of the pseudo-group orbits. Finally, solve the normalization equations (7) with respect to the pseudo-group parameters $g^{(n)}$,

$$g^{(n)} = h^{(n)}(x, u^{(n)}). \tag{8}$$

Once this is done, the nth order moving frame $\rho^{(n)}$ is given by $\rho^{(n)}(x, u^{(n)}) = (x, u^{(n)}, h^{(n)}(x, u^{(n)}))$.

The invariance of the target jet coordinates $(X, \widehat{U}^{(n)})$ under the pseudo-group action and the definition of a moving frame $\rho^{(n)}$ imply

Proposition 1. *The normalized differential invariants*

$$\begin{aligned} H^i(x, u^{(n)}) &= (\rho^{(n)})^*(X^i) = \iota(x^i) = X^i(x, u^{(n)}, h^{(n)}(x, u^{(n)})), \\ I^\alpha_J(x, u^{(n)}) &= (\rho^{(n)})^*\big(U^\alpha_J\big) = \iota\big(u^\alpha_J\big) = \widehat{U}^\alpha_J(x, u^{(n)}, h^{(n)}(x, u^{(n)})), \end{aligned} \tag{9}$$

$i = 1, \ldots, p, \alpha = 1, \ldots, q, 0 \leqslant \#J \leqslant n$, *obtained by replacing the pseudo-group parameters in (6) by (8), form a complete set of functionally independent differential invariants of the nth prolonged pseudo-group action $\mathcal{G}^{(n)}$.*

In (9), r_n of the normalized invariants are constant due to the normalization equations (7). Those invariants are called phantom invariants and the other are referred to as non-phantom invariants.

2.2. *Recurrence formulae*

From the p differential operators (4) and a moving frame $\rho^{(\infty)}$ we derive p independent invariant differential operators

$$\mathcal{D}_i = \sum_{j=1}^{p} \big((\rho^{(\infty)})^*\big(W^j_i\big)\big) D_{x^j}, \qquad i = 1, \ldots, p. \tag{10}$$

Applying the invariant differential operators (10) to the normalized differential invariants (9), with $n = \infty$, gives new differential invariants that can be expressed in terms of the normalized invariants (9) since they constitute a basis of the algebra of differential invariants for the Lie pseudo-group $\mathcal{G}$. Those relations are called recurrence relations. Before writing out the recurrence formulae we recall some facts about the infinitesimal generators of a Lie pseudo-group $\mathcal{G}$. A vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\partial_{x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x, u)\partial_{u^\alpha} \tag{11}$$

is an infinitesimal generator of $\mathcal{G}$ if it is the solution to the infinitesimal determining equations

$$L(x, u, \xi^{(n)}, \phi^{(n)}) = 0, \tag{12}$$

obtained by linearizing the defining equations (2) of the Lie pseudo-group $\mathcal{G}$ at the identity jet $\mathbb{I}_M$. The nth prolongation of the vector field (11) is given by the usual formula,

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i(x, u)\partial_{x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \widehat{\phi}^\alpha_J(x, u^{(n)})\partial_{u^\alpha_J},$$

where

$$\widehat{\phi}^\alpha_J(x, u^{(n)}, \xi^{(n)}, \phi^{(n)}) = D_J\left(\phi^\alpha - \sum_{i=1}^{p} u^\alpha_i \xi^i\right) + \sum_{i=1}^{p} u^\alpha_{J,i}\xi^i. \tag{13}$$

Note that the prolonged coefficients (13) are linear combinations of the derivatives $\xi^i_A, \phi^\alpha_A, \#A \leqslant \#J$. Let

$$\widehat{\psi}^\alpha_J(H, I^{(n)}, \beta^{(n)}, \zeta^{(n)}) = \iota(\widehat{\phi}^\alpha_J(x, u, \xi^{(n)}, \phi^{(n)})),$$

be the invariantization of $\widehat{\phi}^\alpha_J$ obtained by the substitutions

$$x^i \mapsto H^i, \qquad u^\alpha_J \mapsto I^\alpha_J, \qquad \xi^i_A \mapsto \beta^i_A, \qquad \phi^\alpha_A \mapsto \zeta^\alpha_A, \tag{14}$$

where β^i_A and ζ^α_A are the horizontal components of the invariantized Maurer–Cartan forms associated with the Lie pseudo-group $\mathcal{G}$, [2]. Since $\widehat{\phi}^\alpha_J(x, u^{(n)}, \xi^{(n)}, \phi^{(n)})$ are linear in ξ^i_A and ϕ^α_A, $\widehat{\psi}^\alpha_J(H, I^{(n)}, \beta^{(n)}, \zeta^{(n)})$ are linear combinations in the one-forms β^i_A and ζ^α_A. The differential forms β^i_A and ζ^α_A are not linearly independent and remarkably satisfy

Proposition 2. *The one-forms $\beta^i_A, i = 1, \ldots, p$, and $\zeta^\alpha_A, \alpha = 1, \ldots, q, \#A \geqslant 0$, satisfy the linear relations*

$$\mathcal{L}(\ldots, H^i, \ldots, I^\alpha, \ldots, \beta^i_A, \ldots, \zeta^\alpha_A \ldots) = 0, \tag{15}$$

where $\mathcal{L}$ is the completion of the infinitesimal determining equations (12).

The completion $\mathcal{L}$ of L consists of the original equations (12) along with all equations obtained by repeated differentiation. Equations (15) are obtained in two steps, first compute the completion $\mathcal{L}$ then make the substitutions (14). We are now in a position to state

Theorem 1. *The recurrence formulae for the normalized differential invariants (9) are*

$$\sum_{i=1}^{p} (\mathcal{D}_i H^j)\omega^i = \omega^i + \beta^i, \qquad \sum_{i=1}^{p} (\mathcal{D}_i I^\alpha_J)\omega^i = \sum_{i=1}^{p} I^\alpha_{J,i}\omega^i + \widehat{\psi}^\alpha_J, \tag{16}$$

where ω^i are invariant one-forms dual to the invariant differential operators $\mathcal{D}_i$. Their explicit expressions are

$$\omega^i = \sum_{j=1}^{p} ((\rho^{(\infty)})^*(D_{x^j} X^i))dx^j, \qquad i = 1, \ldots, p.$$

The terms β^i and $\widehat{\psi}^\alpha_J$ appearing in (16) are called correction terms.

The recurrence relations for the phantom invariants have their left-hand side equal to zero since these invariants are constant valued. Those equations form a linear system of equations in β^i_A and ζ^α_A which can be solved, if a bona fide cross-section is chosen and the pseudo-group action is locally free at a certain order n. Substituting their expressions in the recurrence relations for the non-phantom invariants gives explicit relations between the invariantly differentiated invariants and the normalized invariants of the form

$$\mathcal{D}_i I^\alpha_J = I^\alpha_{J,i} + R^\alpha_{J,i},$$

where $R^\alpha_{J,i}$ is an expression of the normalized invariants (9).

3. The pseudo-group of equivalence transformations for a differential equation defined by a vector field

In [5], the most general pseudo-group of equivalence transformations for a linear scalar differential equation

$$\sum_{i=1}^{n} a^i(x)\partial_{x^i} u(x) = 0, \qquad x = (x_1, \ldots, x_n), \tag{17}$$

defined by the smooth vector field $\sum_{i=1}^{n} a^i(x)\partial_{x^i}$ is established. The pseudo-group of equivalence transformations consists of all local diffeomorphisms

$$\begin{aligned}&\mathcal{O}_x \times \mathcal{O}_u \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R},\\ &(x, u) \mapsto (X = \psi(x, y), U = \alpha(x, u)),\end{aligned}$$

mapping (17) into an equation of the same form

$$\sum_{i=1}^{n} A^i(X)\partial_{X^i} U(X) = 0,$$

where the functions $A^i(X)$ can be different from the functions $a^i(x)$ appearing in (17). We assume that all the coefficients $a^i(x)$ are nonzero, and also that $n > 1$ because otherwise the equivalence problem is trivial. In this setting, we have

Proposition 3. *The most general pseudo-group of equivalence transformations of (17) consists of all local diffeomorphisms of the form*

$$X^i = \psi_i(x^i), \qquad i = 1, \ldots, n, \tag{18}$$

$$U = u. \tag{19}$$

Under transformation (18) the vector field $\sum_{i=1}^{n} a^i(x)\partial_{x^i}$ is mapped to $\sum_{i=1}^{n} A^i(X)\partial_{X^i}$, with

$$A^i(X) = \psi_i'(x^i)a^i(x), \tag{20}$$

where $\psi_i'(x^i)$ denotes the derivative of $\psi_i(x^i)$ with respect to x^i. From (19) it is clear that the dependent variable u is an invariant of the equivalence problem. We thus ignore this variable in the search of invariant differential functions, and we are thus interested in finding the differential invariants of the Lie pseudo-group

$$\mathcal{G} : X^i = \psi_i(x^i), \qquad A^i = \psi_i'(x^i)a^i, \qquad i = 1, \ldots, n. \tag{21}$$

The defining equations of this Lie pseudo-group are

$$X^i_{x^j} = 0, \qquad \text{if} \quad j \neq i, \qquad A^i = X^i_{x^i} a^i, \qquad i = 1, \ldots, n,$$

where δ^i_j is the Kronecker delta.

For future reference we note that the infinitesimal generator of $\mathcal{G}$ is given by

$$\mathbf{v} = \sum_{i=1}^{n} \xi_i(x^i)\partial_{x^i} + \sum_{i=1}^{n} a^i \xi_i'(x^i)\partial_{a_i}, \tag{22}$$

where $\xi_i(x^i)$ is an arbitrary smooth function of x^i, and $\xi_i'(x^i)$ denotes the derivative of $\xi_i(x^i)$ with respect to x^i.

4. Differential invariants

To find the differential invariants of the Lie pseudo-group (21), we apply the algorithm discussed in section 2.1. We use the multi-index notation $a^i_J = \partial^k a^i/(\partial x^{j_1} \cdots \partial x^{j_k})$, to denote the partial derivatives of the vector field coefficients $a^i, i = 1, \ldots, n$, with respect to the coordinates $x^i, i = 1, \ldots, n$. The prolonged pseudo-group action of $\mathcal{G}$ is found by applying the differential operators

$$D_{X^i} = \frac{1}{\psi_i'(x^i)} D_{x^i}, \qquad i = 1, \ldots, n, \tag{23}$$

where

$$D_{x^j} = \frac{\partial}{\partial x^j} + \sum_{i=1}^{n} \sum_{\#J \geqslant 0} a^i_{J,j} \frac{\partial}{\partial a^i_J}, \qquad j = 1, \ldots, n,$$

to A^i, $i = 1, \ldots, n$. The first few terms are

$$\begin{aligned} A^i_j &= D_{X^j} A^i = \frac{1}{\psi'_j}\left(a^i_j \psi'_i + a^i \delta^i_j \psi''_i\right), \\ A^i_{jk} &= D_{X^k} A^i_j = \frac{1}{\psi'_k}\left[-\frac{\delta^j_k \psi''_j}{(\psi'_j)^2}\left(a^i_j \psi'_i + a^i \delta^i_j \psi''_i\right) \right. \\ &\quad \left. + \frac{1}{\psi'_j}\left(a^i_{jk}\psi'_i + a^i_j \delta^i_k \psi''_i + a^i_k \delta^i_j \psi''_i + a^i \delta^i_j \delta^i_k \psi'''_i\right)\right], \\ &\ \ \vdots \end{aligned} \tag{24}$$

$1 \leqslant i, j, k \leqslant n$. The pseudo-group parameters $\psi_i, \psi'_i, \psi''_i, \ldots$, of the prolonged pseudo-group action are normalized using the cross-section

$$X^i = 0, \qquad A^i = 1, \qquad A^i_{i^k} = 0, \qquad i = 1, \ldots, n, \quad k \geqslant 1, \tag{25}$$

where $A^i_{i^k}$ denotes the kth derivative of A^i with respect to X^i. Solving for the pseudo-group parameters we find

$$\psi^i = 0, \qquad \psi'_i = \frac{1}{a^i}, \qquad \psi''_i = -\frac{a^i_i}{(a^i)^2}, \ldots, \tag{26}$$

$i = 1, \ldots, n$. Replacing expressions (26) in the unnormalized target coordinates of (24), i.e., in A^i_J, $\#J = k \geqslant 1$, with $J = (j_1, \ldots, j_k)$ such that $j_l \neq i$ for some l between 1 and k, we get the differential invariants

$$\begin{aligned} I^i_j &= \iota\left(a^i_j\right) = \frac{a^i_j a^j}{a^i}, \qquad i \neq j, \\ I^i_{jk} &= \iota\left(a^i_{jk}\right) = \frac{a^k}{a^i}\left(\delta^j_k a^j_j\left(a^i_j - \delta^i_j a^i_i\right) + \frac{a^j}{a^i}\left(a^i a^i_{jk} - \delta^i_k a^i_j a^i_i - \delta^i_j a^i_k a^i_i\right)\right), \quad (j,k) \neq (i,i), \\ &\ \ \vdots \end{aligned} \tag{27}$$

The invariants I^i_j, I^i_{jk} of (27) are all functionally independent and give a complete list of invariants for the second prolonged pseudo-group action $\mathcal{G}^{(2)}$.

We now specify the above results to the cases where there are $n = 2$ and $n = 3$ independent variables.

Proposition 4. *Let $\mathcal{N}^n$ be the maximal number of functionally independent differential invariants of orders 1 and 2 in n independent variables.*

(i) For $n = 2$, $\mathcal{N}^2 = 6$, and the invariants are

$$I^i_j = \frac{a^i_j a^j}{a^i}, \qquad I^i_{ij} = a^j a^i_{ij} - \frac{a^j}{a^i} a^i_j a^i_i, \qquad I^i_{jj} = \frac{a^j}{a^i} a^j_j a^i_j + \frac{(a^j)^2}{a^i} a^i_{jj}, \tag{28}$$

with $i, j \in \{1, 2\}$ and $i \neq j$.

(ii) For $n = 3$, $\mathcal{N}^3 = 21$, and the invariants are

$$\begin{aligned} I^i_j &= \frac{a^i_j a^j}{a^i}, & I^i_{ij} &= a^j a^i_{ij} - \frac{a^j}{a^i} a^i_j a^i_i, \\ I^i_{jj} &= \frac{a^j}{a^i} a^j_j a^i_j + \frac{(a^j)^2}{a^i} a^i_{jj}, & I^i_{jk} &= \frac{a^k a^j a^i_{jk}}{a^i}, \end{aligned} \tag{29}$$

with $i, j, k \in \{1, 2, 3\}$, $i \neq j, k$, and $j \neq k$.

Theorem 6 of [5] must be replaced by our proposition 4 above since our list of invariants is more exhaustive. Indeed, for the case $n = 2$, the author of [5] finds four independent differential invariants of orders 1 and 2:

$$T_{ij} = \frac{a^i_j a^j}{a^i}, \qquad K_{ij} = \frac{a^i_{jj} a^j}{a^i_j} + a^j_j,$$

with $i, j \in \{1, 2\}$, and $i \neq j$. Those four invariants are related to the four invariants I^i_j, I^i_{jj} of (28) by the relations

$$I^i_j = T_{ij}, \qquad I^i_{jj} = K_{ij} T_{ij}.$$

So the two new invariants in the list (28) are I^i_{ij}, with $i, j = 1, 2$, and $i \neq j$.

In the case $n = 3$, 15 independent differential invariants of orders 1 and 2 are found in [5]:

$$T_{ij} = \frac{a^i_j a^j}{a^i}, \qquad K_{ij} = \frac{a^i_{jj} a^j}{a^i_j} + a^j_j, \qquad L_{ijk} = a^i_{jk}\left(\frac{a^j a^k}{a^i}\right),$$

with $i, j, k \in \{1, 2, 3\}$, $i \neq j, k$, and $j \neq k$. Those 15 invariants are related to the 15 invariants $I^i_j, I^i_{jj}, I^i_{jk}$ of (29) by the relations

$$I^i_j = T_{ij}, \qquad I^i_{jj} = K_{ij} T_{ij}, \qquad I^i_{jk} = L_{ijk}.$$

So the six new invariants in the list (29) are I^i_{ij}, with $i, j = 1, 2, 3$, and $i \neq j$.

For a general number $n \geqslant 2$ of independent variables, since there are $\binom{n+k-1}{k}$ different kth order derivatives for a scalar function depending on n variables, it follows that there are $\binom{nn+k-1}{k}$ different target coordinates A^i_J, $i = 1, \ldots, n$, with $\#J = k$ in (24). Since our cross-section (25) imposes that $A^i_{i^k} = 0$, $i = 1, \ldots, n$, for $k \geqslant 1$ it follows that there are

$$M^k_n = n\binom{n+k-1}{k} - n, \qquad k \geqslant 1,$$

functionally independent invariants of order k. Hence we have proven.

Proposition 5. *For any value $n \geqslant 2$ of independent variables, the number of functionally independent differential invariants of the second prolongation $\mathcal{G}^{(2)}$ is*

$$\mathcal{N}^n = M^1_n + M^2_n = n(n-1) + \frac{n^2(n+1)}{2} - n = \frac{n^2(n+3)}{2} - 2n,$$

and a basis of such invariants is given by (27).

The conjecture on page 12 of [5] is false, and must be replaced by our proposition 5. As mentioned in the introduction, the conjecture is false since it relies on the wrong number of invariants found by the author for the cases $n = 2, 3$, and the unjustified quantity W_n defined at the top of page 12.

5. Algebra of differential invariants

In this section we give a complete classification of the differential invariants for a generic vector field $\sum_{i=1}^n a^i(x)\partial_{x^i}$.

As discussed in section 2.2, from (23) we obtain n invariant differential operators

$$\mathcal{D}_i = a^i D_{x^i}, \qquad i = 1, \ldots, n, \tag{30}$$

by replacing ψ'_i in (23) by its normalization $\psi'_i = 1/a^i$.

To find the correction terms in the recurrence relations (16) we must first compute the prolongation of the vector field (22). The first terms are

$$\begin{aligned}
&\widehat{\phi}^i_j = a^i_j(\xi'_i - \xi'_j) + \delta^i_j a^i \xi''_i,\\
&\widehat{\phi}^i_{jk} = a^i_{jk}(\xi'_i - \xi'_j - \xi'_k) + \left(\delta^i_k a^i_j + \delta^i_j a^i_k\right)\xi''_i - \delta^j_k a^i_j \xi''_j + \delta^i_j \delta^i_k a^i \xi'''_i,\\
&\vdots
\end{aligned} \tag{31}$$

The correction terms $\widehat{\psi}^i_J = \iota\left(\widehat{\phi}^i_J\right)$ are obtained by making the substitution

$$x^i \mapsto H^i, \qquad a^i_J \mapsto I^i_J$$

$$\xi_i \mapsto \beta^i \qquad \xi'_i \mapsto \beta^i_{X^i} = \beta^i_1, \ldots, \qquad \frac{\mathrm{d}^k \xi_i}{\mathrm{d}(x^i)^k} \mapsto \beta^i_{(X^i)^k} = \beta^i_k, \ldots,$$

$i = 1, \ldots, n$, in (31). We note that the one-forms β^i_k are all functionally independent. This follows from the fact that the functions ξ_i and their derivatives are functionally independent.

Using the recurrence relations for the phantom invariants $\iota(x^i) = 0$, $\iota(A^i) = 1$, $\iota\left(A^i_{i^k}\right) = 0$, $k \geqslant 1$, $i = 1, \ldots, n$, we find the explicit expressions for the one-forms β^i_k:

$$\begin{aligned}
&0 = \omega^i + \beta^i && \Rightarrow && \beta^i = -\omega^i,\\
&0 = \sum_{j \neq i} I^i_j \omega^j + \beta^i_1, && \Rightarrow && \beta^i_1 = -\sum_{j\neq i} I^i_j \omega^j,\\
&0 = \sum_{j \neq i} I^i_{ij} \omega^j + \beta^i_2, && \Rightarrow && \beta^i_2 = -\sum_{j\neq i} I^i_{ij} \omega^j,\\
&\quad\vdots\\
&0 = \sum_{j \neq i} I^i_{i^{k-1}j} \omega^j + \beta^i_k, && \Rightarrow && \beta^i_k = -\sum_{j\neq i} I^i_{i^{k-1}j} \omega^j,
\end{aligned} \tag{32}$$

$i = 1, \ldots, n$, where $I^i_{i^k j}$ is the invariant $I^i_{i,i,\ldots,i,j}$ with ki's as subscripts.

Substituting expressions (32) in the recurrence relations for the first-order non-phantom invariants we find

$$\sum_{i=1}^n \mathcal{D}_i I^j_k \omega^i = \sum_{i=1}^n I^j_{ki}\omega^i + I^j_k \left(\sum_{i \neq j} I^j_i \omega^i - \sum_{i \neq k} I^k_i \omega^i \right), \qquad j \neq k.$$

By induction on the order of the prolonged vector field coefficients (31) we see that the correction terms $\widehat{\psi}^i_J$ with at least one $j_l \neq i$ involves only terms in β^i_j with $j \leqslant \#J - 1$. From (32), we conclude that the correction terms for the recurrence relations of the non-phantom invariants of order $\#J \geqslant 1$ depend on non-phantom invariants of order at most $\#J$. Hence any normalized invariant $I^i_{J,j}$ of order $\#J + 1$ can be written as

$$I^i_{J,j} = \mathcal{D}_j I^i_J + R^i_{J,j}, \qquad \#J \geqslant 1, \qquad 1 \leqslant i, j \leqslant n,$$

where $R^i_{J,j}$ depends on normalized invariants of order at most $\#J$. Base on those considerations we conclude that the first-order differential invariants I^i_j, $i, j = 1, \ldots, n$, $i \neq j$, generate the algebra of differential invariants of the equivalence pseudo-group (21).

For a generic vector field $\sum_{i=1}^n a^i(x)\partial_{x^i}$, the number of first-order differential invariants generating the algebra of differential invariants can be greatly reduced using the commutation relations between the invariant differential operators (30). By direct computation

$$[\mathcal{D}_j, \mathcal{D}_k] = \left[a^j D_{x^j}, a^k D_x^k\right] = I_j^k \mathcal{D}_k - I_k^j \mathcal{D}_j. \tag{33}$$

Under the assumption that the two normalized invariants $I_{j_o}^{i_o}$, and $I_{i_o}^{j_o}$, $i_o \neq j_o$, with i_o, j_o fixed, satisfy

$$\det \begin{pmatrix} \mathcal{D}_l I_{j_o}^{i_o} & \mathcal{D}_k I_{j_o}^{i_o} \\ \mathcal{D}_l I_{i_o}^{j_o} & \mathcal{D}_k I_{i_o}^{j_o} \end{pmatrix} \neq 0, \tag{34}$$

for all $(k, l) \neq (i_o, j_o)$ or $(k, l) \neq (j_o, i_o)$, $k \neq l$, we can reduce the generating set of invariants to $I_{j_o}^{i_o}$ and $I_{i_o}^{j_o}$. Indeed the assumption (34) implies that we can solve the linear system

$$\begin{pmatrix} [\mathcal{D}_k, \mathcal{D}_l] I_{j_o}^{i_o} \\ [\mathcal{D}_k, \mathcal{D}_l] I_{i_o}^{j_o} \end{pmatrix} = \begin{pmatrix} \mathcal{D}_l I_{j_o}^{i_o} & \mathcal{D}_k I_{j_o}^{i_o} \\ \mathcal{D}_l I_{i_o}^{j_o} & \mathcal{D}_k I_{i_o}^{j_o} \end{pmatrix} \begin{pmatrix} I_k^l \\ -I_l^k \end{pmatrix},$$

$(k, l) \neq (i, j)$ or $(k, l) \neq (j, i)$, $k \neq l$, for I_l^k and I_k^l in terms of $I_{j_o}^{i_o}$, $I_{i_o}^{j_o}$ and their invariant derivatives. An explicit computation of the determinants appearing in (34) using (30) and (27) confirms that the determinants are not identically zero for a generic vector field $\sum_{i=1}^n a^i(x)\partial_{x^i}$.

If furthermore

$$\mathcal{D}_{j_o} I_{j_o}^{i_o} \neq 0, \tag{35}$$

which holds for a generic vector field, we can use the commutation relation (33) for $\mathcal{D}_{i_o}$ and $\mathcal{D}_{j_o}$ to write

$$I_{i_o}^{j_o} = \frac{1}{\mathcal{D}_{j_o} I_{j_o}^{i_o}} \left(\left[\mathcal{D}_{i_o}, \mathcal{D}_{j_o}\right] I_{j_o}^{i_o} + I_{j_o}^{i_o} \mathcal{D}_{i_o} I_{j_o}^{i_o} \right).$$

Hence for a generic vector field $\sum_{i=1}^n a^i(x)\partial_{x^i}$, we conclude that all differential invariants of the equivalence pseudo-group (21) can be expressed in terms of the single invariant $I_{j_o}^{i_o}$, $i_o \neq j_o$, and its invariantly differentiated consequences $\mathcal{D}_{j_1} \cdots \mathcal{D}_{j_k} I_{j_o}^{i_o}$, $1 \leqslant j_1, \ldots, j_k \leqslant n$, $k \geqslant 1$.

For non-generic vector fields , some of the non-degeneracy conditions (34) and (35) might not hold. The problem then splits in many different sub-cases, depending on which determinants in (34) are identically zero and if the assumption (35) holds. But in most sub-cases we can still use the commutator relations (33) to reduce the generating set of first-order differential invariants $\left\{I_j^i : i \neq j\right\}$ to a subset of itself.

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